方阵的正整数次幂： $\forall A \in F^{n\times n}$

$$A^k := \underbrace{A \cdots A}_{k个} \qquad k=1,2,\cdots$$

$$A^0 := I$$

例：Fibonacci 数列通项公式

$$\left(F_1 = F_2 = 1 ,\quad F_n = F_{n-1} + F_{n-2} \quad (n\geq 3) \right)$$

证：$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$ $\square$

矩阵多项式 $\forall f(x) = c_0 + c_1 x + \cdots + c_k x^k \in F[x]$

$$f(A) := c_0 I_{(n)} + c_1 A + \cdots + c_k A^k \in F^{n\times n}$$

$$\begin{cases} (A+B)^2 = A^2 + 2AB + B^2 \quad ? \\ (I+A)^n = \sum_{k=0}^{n} \binom{n}{k} A^k \quad ? \\ f(A)\,g(A) = g(A)\,f(A) \quad ? \end{cases}$$

①

例: $J_n = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}_{n\times n} \Rightarrow J_n^2 = ?$

$J_n^3 = ? \cdots$

$n = 2, 3, 4$

$$\left(aI_{(n)} + bJ_n\right)^k = ?$$

J 换成其它矩阵，例如 $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$?

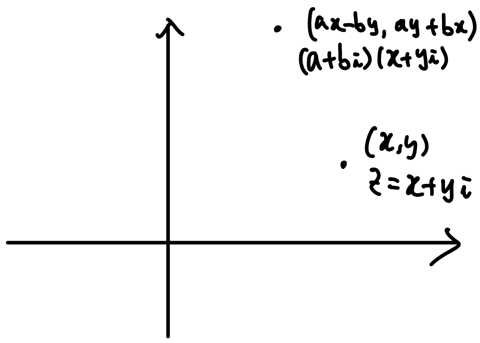
复数与二阶矩阵

$$f: \mathbb{C} \longrightarrow F^{n\times n}$$

$$a+bi \longmapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

性质：f 保持加法与乘法，即 $\forall\ z_1 = a+bi,\ z_2 = c+di$

- $f(z_1+z_2) = f(z_1) + f(z_2)$
- $f(z_1 z_2) = f(z_1)\cdot f(z_2)$



$$\begin{array}{ccc} (x+yi) & \xrightarrow{a+bi} & (ax-by)+(ay+bx)i \\ \Updownarrow & & \Updownarrow \\ (x,y) & \xrightarrow{a+bi} & (ax-by,\ ay+bx)=(x,y)\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{array}$$

坐标变换：$[O;\vec{e_1},\vec{e_2},\vec{e_3}]$，$[O',\vec{e_1'},\vec{e_2'},\vec{e_3'}]$

设空间中的点 P 在 $[O;\vec{e_1},\vec{e_2},\vec{e_3}]$ 下的坐标为 (x,y,z)

在 $[O';\vec{e_1'},\vec{e_2'},\vec{e_3'}]$ 下的坐标为 (x',y',z').

(x,y,z) 与 (x',y',z') 之间的关系？

设 O 在 $[O';\vec{e_1'},\vec{e_2'},\vec{e_3'}]$ 下的坐标为 (x_0',y_0',z_0')，即

$$\overrightarrow{O'O} = (\vec{e_1'},\vec{e_2'},\vec{e_3'})\begin{pmatrix} x_0' \\ y_0' \\ z_0' \end{pmatrix} \leftarrow X_0'$$

设 $\vec{e_j}$ 在 $[O';\vec{e_1'},\vec{e_2'},\vec{e_3'}]$ 下的坐标为 (a_{1j},a_{2j},a_{3j})，即

$$(\vec{e_1},\vec{e_2},\vec{e_3}) = (\vec{e_1'},\vec{e_2'},\vec{e_3'})\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \leftarrow A$$

$$\begin{cases} \overrightarrow{OP} = (\vec{e_1},\vec{e_2},\vec{e_3})\begin{pmatrix} x \\ y \\ z \end{pmatrix} \leftarrow X \\ \overrightarrow{O'P} = (\vec{e_1'},\vec{e_2'},\vec{e_3'})\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \leftarrow X' \end{cases}$$

$$\Rightarrow \quad \overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P} = (\vec{e}_1', \vec{e}_2', \vec{e}_3') \begin{pmatrix} x_0' \\ y_0' \\ z_0' \end{pmatrix} + (\vec{e}_1', \vec{e}_2', \vec{e}_3') \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$(\vec{e}_1', \vec{e}_2', \vec{e}_3') X' = (\vec{e}_1', \vec{e}_2', \vec{e}_3') \left(AX + X_0' \right)$$

$$\Rightarrow \quad X' = AX + X_0'$$

问题 给定 X' 如何计算 X ？ $b := X' - X_0'$

$\Rightarrow \quad AX = b$ 求解 X ！

例：线性方程组 (3.1)

$$Ax = b$$

其中 $A = (a_{ij})_{m \times n}$， $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

§ 逆矩阵

定义：$A \in F^{n\times n}$ 为 n 阶方阵. 如果存在 n 阶方阵 X 满足

$$XA = I = AX$$

则称 A 可逆，并称 X 为 A 的逆矩阵，记作 A^{-1}

非奇异方阵，奇异方阵

性质（存在则唯一）：X, Y 为 A 的逆矩阵，则 $X=Y$.

证：$X = X\cdot I = X(AY) = (XA)\cdot Y = I\cdot Y = Y$ □

性质：1) $(A^{-1})^{-1} = A$

2) $(\lambda A)^{-1} = \lambda^{-1}\cdot A^{-1}$ $(\lambda \neq 0)$

3) $(AB)^{-1} = B^{-1}\cdot A^{-1}$

证：...

例：(1) $ad \neq bc$，则 i.e. $(a,b) \not\parallel (c,d)$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

(2) $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)$, $\vec{c}=(c_1,c_2,c_3)$ 则

当 $(\vec{a}\times\vec{b})\cdot\vec{c}\neq 0$ 时,

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}^{-1} = \frac{1}{(a\times b)\cdot c}\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$$

其中 $\vec{u}=(u_1,u_2,u_3)=\vec{b}\times\vec{c}$

$\vec{v}=(v_1,v_2,v_3)=\vec{c}\times\vec{a}$

$\vec{w}=(w_1,w_2,w_3)=\vec{a}\times\vec{b}$